

Log-Lipschitz continuity of the vector field on the attractor of certain parabolic equations

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Abstract

We discuss various issues related to the finite-dimensionality of the asymptotic dynamics of solutions of parabolic equations. In particular, we study the regularity of the vector field on the global attractor associated with these equations. We show that certain dissipative partial differential equations possess a linear term that is log-Lipschitz continuous on the attractor. We then prove that this property implies that the associated global attractor \mathcal{A} lies within a small neighbourhood of a smooth manifold, given as a Lipschitz graph over a finite number of Fourier modes. Consequently, the global attractor \mathcal{A} has zero Lipschitz deviation and, therefore, there are linear maps L into finite-dimensional spaces, whose inverses restricted to $L\mathcal{A}$ are Hölder continuous with an exponent arbitrarily close to one.

1 Introduction

The existence of global attractors with finite upper box-counting dimension for a wide class of dissipative equations (see Babin and Vishik [2], Foias and Temam [14], Hale [15], Temam [35], for example) strongly suggests that it might be possible to construct a system of ordinary differential equations whose asymptotic dynamics reproduces the dynamics on the original attractor. However, because of the complexity of the flow on the attractor \mathcal{A} and

its irregular structure, the finite dimensionality of \mathcal{A} alone is not immediately sufficient to guarantee the existence of such a system of ordinary differential equations.

Indeed, the existence of an ordinary differential equation with analogous asymptotic dynamics has only been proved for dissipative partial differential equations that possess an inertial manifold, i.e. a finite-dimensional, positively invariant Lipschitz manifold that attracts all orbits exponentially (see Constantin and Foias [6], Constantin *et al.* [7], Foias, Manley and Temam [11], Foias, Sell and Temam [13], Temam [35], for more details). All the methods available in the literature construct inertial manifolds as graphs of functions from a finite-dimensional eigenspace associated with the low Fourier modes into the complementary infinite-dimensional eigenspace corresponding to the high Fourier modes.

Foias *et al.* [12] showed that if a ‘certain spectral gap condition’ holds for a given system, then it will possess an inertial manifold. Unfortunately, this sufficient condition is quite restrictive, and there are many equations, such as the 2D Navier-Stokes equations, that do not satisfy it. Nonetheless, Kukavica [18] and [19] showed that the global attractor of certain dissipative equations, such as the Burgers equation in one space dimension, lies in a Lipschitz graph over a finite number of Fourier modes independently of the theory of inertial manifolds.

Romanov [33] discussed the problem of a finite-dimensional description of the asymptotic behaviour of dissipative equations more abstractly. He defined the dynamics on the attractor \mathcal{A} to be ‘finite-dimensional’ if there exists a bi-Lipschitz map $\Pi : \mathcal{A} \rightarrow \mathbb{R}^N$, for some N , and an ordinary differential equation with a Lipschitz vector field on \mathbb{R}^N such that the dynamics on \mathcal{A} and $\Pi(\mathcal{A})$ are conjugated under Π . He then showed that this property is equivalent to the attractor being contained in a finite-dimensional Lipschitz manifold, given as a graph over a sufficiently large number of Fourier modes. Hence, his definition and that of an inertial manifold are much more similar than they first appear. In Section 3, we investigate other possible ways to define when the asymptotic dynamics of solutions of parabolic equations are ‘finite-dimensional’. We discuss conditions under which an attractor is a subset of a Lipschitz manifold given as a graph over a finite-dimensional space; in particular, we give a concise proof of an important part of Romanov’s result.

To illustrate the problem of constructing a finite set of ordinary differential equations that reproduces the dynamics on the global attractor, consider

a governing equation $\dot{u} = \mathcal{G}(u)$ defined on a Hilbert space H . Suppose there exists a linear map $L : H \rightarrow \mathbb{R}^N$ that is injective on \mathcal{A} . In order to study the smoothness of the embedded equation on $X = L\mathcal{A}$,

$$\dot{x} = h(x) = L\mathcal{G}L^{-1}(x), \quad x \in X, \quad (1)$$

one needs to consider the continuity of the vector field on \mathcal{A} and the continuity of the inverse of the embedding L restricted to X .

The regularity of the embedding L has been discussed in a variety of papers (see Mañé [22], Ben-Artzi *et al.* [4], Eden *et al.* [9], Foias and Olson [10], Hunt and Kaloshin [17], Olson and Robinson [27], Robinson [32] for more details). Hunt and Kaloshin [17], for example, showed that if \mathcal{A} has finite upper box-counting dimension, there exists a linear map $L : H \rightarrow \mathbb{R}^N$ that is injective on \mathcal{A} and whose inverse $L^{-1} : X \rightarrow \mathcal{A}$ is Hölder continuous with exponent α , i.e. there exists $C > 0$ such that

$$C\|L(u) - L(v)\|^\alpha \geq \|u - v\|, \quad \text{for all } u, v \in \mathcal{A}. \quad (2)$$

Introduced by Assouad [1], the Assouad dimension is another useful notion of dimension in the study of embeddings of finite-dimensional sets. The *Assouad dimension* $\dim_A(X)$ of X can be defined as the infimum over all d for which there exists a constant K such that

$$\mathcal{N}(r, \rho) \leq K(r/\rho)^d \quad \text{for } 0 < \rho < r < 1,$$

where $\mathcal{N}(r, \rho)$ is the number of ρ -balls required to cover any r -ball in \mathcal{A} (for proof see Olson [26, Theorem 2.3]). For a comprehensive treatment of the Assouad dimension, see Luukkainen [21].

The strongest existing embedding result, due to Robinson and Olson [27, Theorem 5.6], guarantees the existence of an embedding $L : H \rightarrow \mathbb{R}^N$ such that L^{-1} is γ -log-Lipschitz, i.e. there exist $\gamma \geq 0$ and $C > 0$ such that

$$\|L^{-1}(x) - L^{-1}(y)\| \leq C\|x - y\| \left(\log \frac{M}{\|x - y\|} \right)^\gamma, \quad \text{for all } x, y \in X,$$

where M is a constant depending on X , if the set $X - X$ of differences between elements of X has Assouad dimension $\dim_A(X - X) < s < N$. However, there is no general method to bound the Assouad dimension of global attractors associated with dissipative equations.

In this paper, we will focus our discussion on the regularity of the vector field \mathcal{G} in (1). If one would like a system of ordinary differential equations with unique solutions that generates a flow $\{S_t\}$, then the embedded vector field h in X does not need to be Lipschitz; it is sufficient for h to be 1-log-Lipschitz¹. Hence, one needs to show that there exist

- (i) an exponent $\eta > 0$ such that the vector field on the attractor \mathcal{A} is η -log-Lipschitz in H , and
- (ii) an exponent $\gamma > 0$ such the inverse of linear embedding $L : H \rightarrow \mathbb{R}^N$ is γ -log-Lipschitz when restricted to X ,

for which the inequality $\eta + \gamma \leq 1$ holds so that the solutions are unique.

It is, therefore, reasonable to consider separately the problem of the regularity of the vector field on the global attractor associated with certain parabolic equations. If we assume the very strong condition that L is a bi-Lipschitz embedding, then we would only need the vector field to be 1-log-Lipschitz to guarantee existence and uniqueness of solutions of the embedded equation. In Section 4, we show that certain dissipative partial differential equations, such as the 2D Navier-Stokes equations, possess a linear term that is 1-log-Lipschitz continuous using methods developed by Kukavica [20].

In Section 5, we prove that the 1-log-Lipschitz continuity of the linear term implies that there exists a family of Lipschitz manifolds \mathcal{M}_N such that the distance between the N -dimensional manifold \mathcal{M}_N and the attractor \mathcal{A} is exponentially small in N . (It is interesting to note that this result does not rely explicitly on the fact that the solutions of these semilinear equations satisfy the geometric ‘squeezing property’, introduced by Foias and Temam [14] and on which many other constructions depend, eg. Foias *et al.* [11] or Pinto de Moura and Robinson [28]).

In Section 6, we show that, for certain dissipative equations, one can make the Hölder exponent in (2) as close to one as required by taking N sufficiently large.

¹It is, then, possible to extend $h : X \rightarrow \mathbb{R}^N$ to 1-log-Lipschitz function $\mathcal{H} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ (see McShane [23] for details)

2 Notation and general setting

Consider a dissipative parabolic equation written as an evolution equation of the form

$$\frac{du}{dt} + Au = F(u) \quad (3)$$

in a separable real Hilbert space H with scalar product (\cdot, \cdot) and norm $\|\cdot\|$. We suppose that A is a positive self-adjoint linear operator with compact inverse and dense domain $D_H(A) \subset H$. For each $\alpha \geq 0$, we denote by $D_H(A^\alpha)$ the domain of A^α , i.e.

$$D_H(A^\alpha) = \{u : A^\alpha u \in H\};$$

these are Hilbert spaces with inner product $(u, v)_\alpha = (A^\alpha u, A^\alpha v)$ and norm $\|u\|_\alpha = \|A^\alpha u\|$. We know that for $\alpha > \beta$, the embedding $D_H(A^\alpha) \subset D_H(A^\beta)$ is dense and continuous such that

$$\|u\|_\beta \leq \tilde{C}(\alpha, \beta) \|u\|_\alpha, \quad \text{for } u \in D_H(A^\alpha) \quad (4)$$

(see Sell and You [34], for details). Moreover, we assume that, for some $0 \leq \alpha \leq 1/2$, the nonlinear term F is locally Lipschitz from $D_H(A^\alpha)$ into H , for $u, v \in D_H(A^\alpha)$,

$$\|F(u) - F(v)\| \leq K(R) \|A^\alpha(u - v)\|, \quad \text{with } \|A^\alpha u\|, \|A^\alpha v\| \leq R, \quad (5)$$

where K is a constant depending only on R . This abstract setting includes, among others, the 2D Navier-Stokes equations and the original Burgers equation with Dirichlet boundary values (see Eden *et al.* [9], Temam [35] for example).

Since A is self-adjoint and its inverse is compact, H has an orthonormal basis $\{w_j\}_{j \in \mathbb{N}}$ consisting of eigenfunctions of A such that

$$Aw_j = \lambda_j w_j \quad \text{for all } j \in \mathbb{N}$$

with $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$. With $n \in \mathbb{N}$ fixed, define the finite-dimensional orthogonal projections P_n and their orthogonal complements Q_n by

$$P_n u = \sum_{j=1}^n (u, w_j) w_j \quad \text{and} \quad Q_n u = \sum_{j=n+1}^{\infty} (u, w_j) w_j.$$

Hence, we can write $u = P_n u + Q_n u$, for all $u \in H$. The orthogonal projections P_n and Q_n are bounded on the Hilbert spaces $D_H(A^\alpha)$, for any $\alpha > 0$ (see (4)). Notice that $P_n H = P_n D_H(A^\alpha) \subset D_H(A^\alpha)$, since $P_n H$ is a finite-dimensional subspace generated by the eigenvectors of A corresponding to the first n eigenvalues of A . These spectral projections commute with the operators e^{-At} for $t > 0$, i.e., $P_n e^{-At} = e^{-At} P_n$ and $Q_n e^{-At} = e^{-At} Q_n$. Moreover, we have the following estimate

$$\|e^{-At} Q_n u\|_\alpha \leq \sup_{j \geq n+1} \{\lambda_j^\alpha e^{-\lambda_j t}\} \|Q_n u\| \leq b_{n,\alpha}(t) \|Q_n u\|,$$

where

$$b_{n,\alpha}(t) = \begin{cases} \left(\frac{et}{\alpha}\right)^{-\alpha}, & \text{for } 0 < t \leq \alpha/\lambda_{n+1} \\ \lambda_{n+1}^\alpha e^{-\lambda_{n+1} t}, & \text{for } t \geq \alpha/\lambda_{n+1} \end{cases}$$

Therefore,

$$\left\| A^\alpha e^{-At} Q_n \right\|_{\mathcal{L}(H,H)} \leq b_{n,\alpha}(t). \quad (6)$$

Within this general setting, one can prove the local existence and uniqueness of solutions of (3) (see Henry [16] for details). In particular, it follows from Henry [16, Lemma 3.3.2] that the solution of the nonlinear equation (3) with initial condition $u(t_0) = u_0$, $t > t_0$ is given by the variation of constants formula

$$u(t) = e^{-A(t-t_0)} u_0 + \int_{t_0}^t e^{-A(t-s)} F(u(s)) \, ds, \quad (7)$$

for $t > t_0$ and $u(t_0) \in D_H(A^\alpha)$.

Thus, we can define $\{\Phi_t\}_{t \geq 0}$ to be the semigroup in $D_H(A^\alpha)$ generated by (3) such that, for any initial condition $u_0 \in D_H(A^\alpha)$, there exists a unique solution given by $u(t; u_0) = \Phi_t u_0$. We assume that this system is dissipative, i.e. that there exists a compact invariant absorbing set. It follows from standard results that (3) possesses a global attractor \mathcal{A} , the maximal compact invariant set in $D_H(A^\alpha)$ that uniformly attracts the orbits of all bounded sets (see Babin and Vishik [2], Hale [15], Robinson [30], Temam [35]). So, if $u(0) = u_0 \in \mathcal{A}$, then there is a unique solution $u(t) = \Phi_t u_0 \in \mathcal{A}$ that is defined for all $t \in \mathbb{R}$.

3 Finite-dimensionality of flows

Inertial manifolds, as discussed in the Introduction, are a convenient, although indirect, method to obtain a system of ordinary differential equations that reproduces the asymptotic dynamics on the global attractor. Foias, Sell and Temam [12] showed that if a certain ‘spectral gap condition’ holds – if there exists an n such that $\lambda_{n+1} - \lambda_n > k\lambda_{n+1}^\alpha$, where k is a constant depending on F – then the system (3) possesses an inertial manifold \mathcal{M} . Unfortunately, this condition is very restrictive and there are many equations, such as the 2D Navier-Stokes equations, that do not satisfy it.

Romanov considered in [33] a more general definition of what it means for a system to be asymptotically finite-dimensional. We will see that this definition implies the existence of a Lipschitz manifold that contains the attractor, but does not require it to be exponentially attracting. Romanov defined the dynamics on a global attractor \mathcal{A} to be *finite-dimensional* if for some $N \geq 1$ there exist:

- (i) an ordinary differential equation $\dot{x} = \mathcal{H}(x)$ with a Lipschitz vector field $\mathcal{H}(x)$ in \mathbb{R}^N ,
- (ii) a corresponding flow $\{S_t\}$ on \mathbb{R}^N and
- (iii) a bi-Lipschitz embedding $\Pi : \mathcal{A} \rightarrow \mathbb{R}^N$, such that $\Pi(\Phi_t u) = S_t \Pi(u)$ for any $u \in \mathcal{A}$ and $t \geq 0$.

It follows from this definition that the evolution operators Φ_t are injective on \mathcal{A} for $t > 0$. If we set $\Phi_{-t} = \Pi^{-1} S_{-t} \Pi$, then we see that in fact Φ_t is Lipschitz on \mathcal{A} even for $t < 0$. Hence, we obtain a Lipschitz flow $\{\Phi_t\}$ defined on \mathcal{A} for all $t \in \mathbb{R}$. In particular, there exist $C \geq 1$ and $\mu > 0$ such that

$$\|\Phi_t u - \Phi_t v\|_\alpha \leq C \|u - v\|_\alpha e^{\mu|t|}, \quad (8)$$

for every $t \in \mathbb{R}$.

Considering the general Banach space case, Romanov [33] proved that the finite-dimensionality of the dynamics on the attractor \mathcal{A} is equivalent to five different criteria. In this paper, however, we are only interested in the consequences of the finite-dimensionality of the dynamics on \mathcal{A} . Since our setting is simpler than Romanov’s [33], the arguments involved in the proof become more transparent. Hence, we include here a concise and self-contained proof that if the attractor \mathcal{A} has ‘finite-dimensional dynamics’ in

Romanov's sense, then it must lie on a finite-dimensional manifold, defined as the graph of a Lipschitz function over $P_n H$, for some $n < \infty$.

Theorem 3.1. (Romanov [33]) *If the dynamics on \mathcal{A} is finite-dimensional, then, for some $n \geq 1$, there exists a finite-dimensional projection P_n such that*

$$\|u - v\|_\alpha \leq c \|P_n(u - v)\|_\alpha \quad \text{for all } u, v \in \mathcal{A}, \quad (9)$$

where $c = c(\mathcal{A}, P_n)$.

Proof. Consider the variation of constants formula (7) with $t = 0$ and $u(0) = u \in \mathcal{A}$. If we apply the projection operator Q_n to both sides of (7), then

$$Q_n u = Q_n e^{At_0} u(t_0) + \int_{t_0}^0 e^{As} Q_n F(u(s)) \, ds.$$

Now, since the compact set \mathcal{A} is bounded in $D_H(A^\alpha)$ and $u(t) \in \mathcal{A}$, it follows from (6) that $\lim_{t_0 \rightarrow -\infty} \|Q_n e^{At_0} u(t_0)\|_\alpha = 0$. Consequently, letting t_0 tend to $-\infty$ we obtain

$$Q_n u = \int_{-\infty}^0 e^{As} Q_n F(\Phi_s u) \, ds,$$

which converges in $D_H(A^\alpha)$. It follows from (8) that, for $u, v \in \mathcal{A}$,

$$\begin{aligned} \|Q_n u - Q_n v\|_\alpha &\leq \int_{-\infty}^0 \left\| e^{As} Q_n (F(\Phi_s u) - F(\Phi_s v)) \right\|_\alpha \, ds \\ &\leq K \int_{-\infty}^0 \left\| A^\alpha e^{As} Q_n \right\|_{op} \|\Phi_s u - \Phi_s v\|_\alpha \, ds \\ &\leq KC \|u - v\|_\alpha \int_{-\infty}^0 \left\| A^\alpha e^{As} Q_n \right\|_{op} e^{\mu s} \, ds \end{aligned}$$

Using estimate (6) with $t = -s$, we find that

$$\|Q_n u - Q_n v\|_\alpha \leq KC \|u - v\|_\alpha \int_{-\infty}^0 b_{n,\alpha}(s) e^{\mu s} \, ds$$

from which the inequality

$$\|Q_n u - Q_n v\|_\alpha \leq \vartheta_n \|u - v\|_\alpha, \quad (10)$$

where

$$\vartheta_n := \frac{1}{KC} \left\{ \left(\frac{e}{\alpha} \right)^{-\alpha} \left(\frac{\alpha}{\lambda_{n+1}} \right)^{1-\alpha} \frac{1}{1-\alpha} + \frac{\lambda_{n+1}^\alpha}{\lambda_{n+1} + \mu} e^{\frac{-\alpha(\lambda_{n+1} + \mu)s}{\lambda_{n+1}}} \right\},$$

can be obtained by simple algebraic manipulation.

Note that, since $0 < \alpha < 1$ and λ_{n+1} tends to infinity as $n \rightarrow \infty$, one can choose n sufficiently large to ensure that $\vartheta_n < 1$. Since $P_n + Q_n = I$, it follows that

$$\begin{aligned} \|Q_n(u_0 - v_0)\|_\alpha &\leq \vartheta_n \|P_n(u_0 - v_0)\|_\alpha + \vartheta_n \|Q_n(u_0 - v_0)\|_\alpha \\ &\leq \frac{\vartheta_n}{1 - \vartheta_n} \|P_n(u_0 - v_0)\|_\alpha. \end{aligned}$$

Thus,

$$\|u_0 - v_0\|_\alpha \leq \frac{1}{1 - \vartheta_n} \|P_n(u_0 - v_0)\|_\alpha$$

and hence the attractor is contained in the graph of a Lipschitz function $\phi : P_n H \rightarrow Q_n H$. \square

Under the assumption that the non-linear term F is in $C^2(D_H(A^\alpha), H)$, Romanov [33] showed that the finite-dimensionality of \mathcal{A} implies that the vector field $\mathcal{G}(u) = -Au + F(u)$ is Lipschitz². However, it is not clear how to adapt his argument to prove that A is Lipschitz. Here we give a simple argument that shows that finite-dimensionality implies that the operator A^β is Lipschitz in \mathcal{A} , provided that $\alpha + \beta < 1$.

Corollary 3.2. *If the dynamics on \mathcal{A} is finite-dimensional, then, for β with $\alpha + \beta < 1$, A^β is Lipschitz on \mathcal{A} , i.e.*

$$\|A^\beta(u - v)\|_\alpha \leq M \|u - v\|_\alpha, \quad \text{for all } u, v \in \mathcal{A},$$

where α is given by (5).

²If $F \in C^2(D_H(A^\alpha), H)$, then it follows from Henry [16, Corollary 3.4.6] that the map $(u_0, t) \mapsto u(t)$ is also in $C^2(\mathbb{R}^+ \times D_H(A^\alpha), D_H(A^\alpha))$. Hence, the function $(u_0, t) \mapsto du(t)/dt$ is C^1 with respect to (u_0, t) . Since $du(t)/dt = \mathcal{G}(u(t))$, for a fixed time (we choose $t = 1$), the map $u_0 \mapsto \mathcal{G}(u(1))$ is also a C^1 -function and, consequently, a Lipschitz function. The finite dimensionality of the dynamics on \mathcal{A} implies that the map $u_0 \mapsto u(1)$ is bi-Lipschitz on \mathcal{A} . And, therefore, the map $u(1) \mapsto \mathcal{G}(u(1))$ is Lipschitz continuous.

Proof. It follows from (8) that

$$\|u(t) - v(t)\|_\alpha \leq Ce^{\mu|t|} \|u_0 - v_0\|_\alpha \quad \text{for all } u_0, v_0 \in D_H(A^\alpha).$$

Let β be such that $\alpha + \beta < 1$. Then,

$$\begin{aligned} \|A^\beta(u(t) - v(t))\|_\alpha &\leq \|A^\beta e^{-At}\|_{\text{op}} \|u_0 - v_0\|_\alpha \\ &\quad + KC \|u_0 - v_0\|_\alpha \int_0^t \|A^{\alpha+\beta} e^{-A(t-s)}\|_{\text{op}} e^{\mu|s|} ds \\ &\leq \overline{M} \|u_0 - v_0\|_\alpha \leq M \|u(t) - v(t)\|_\alpha e^{\mu|s|} \end{aligned}$$

Since \mathcal{A} is invariant, given $u, v \in \mathcal{A}$, we have that $u = S(t)u_0$ and $v = S(t)v_0$, for some $u_0, v_0 \in \mathcal{A}$. Hence,

$$\|A^\beta(u - v)\|_\alpha \leq M \|u - v\|_\alpha, \quad \text{for all } u, v \in \mathcal{A}.$$

□

Although Romanov's result establishes an important criterion for the 'ideal' definition of finite-dimensionality of the dynamics on the attractor, to require \mathcal{A} to admit a bi-Lipschitz embedding into some \mathbb{R}^N is very strong and unlikely to be satisfied in general. A sensible way to weaken this definition would be to relax the bi-Lipschitz assumption and assume the embedded vector field \mathcal{H} to be just log-Lipschitz. However, the argument used in the proof of Theorem 3.1 would, then, not work. Hence, it is more reasonable to remove the assumption that the flow is generated by an ODE and define the following:

Definition 3.3. *The dynamics on a global attractor \mathcal{A} is finite-dimensional if, for some $N \geq 1$, there exist an embedding $\Pi : \mathcal{A} \rightarrow \mathbb{R}^N$ that is injective on \mathcal{A} , a flow $\{S_t\}$ in \mathbb{R}^N and a global attractor X , such that the dynamics on \mathcal{A} and X are conjugate under Φ via $\Pi(\Phi_t u) = S_t \Pi(u)$, for any $u \in \mathcal{A}$ and $t \geq 0$.*

However, even in this weak sense, it is still an open problem whether the finite-dimensionality of the global attractor \mathcal{A} implies that the dynamics on \mathcal{A} is finite-dimensional.

4 Log-Lipschitz continuity of the vector field

In the last section, we showed that if the dynamics on the attractor is finite-dimensional, then A^β is Lipschitz on \mathcal{A} provided that $\alpha + \beta < 1$, where α is given by (5). It is relatively easy to show that the converse is also true (see Robinson [31]).

Proposition 4.1. *Suppose that A^β is Lipschitz continuous on the attractor from $D_H(A^\alpha)$ into itself, i.e.*

$$\|A^\beta u - A^\beta v\|_\alpha \leq M \|u - v\|_\alpha \quad \text{for all } u, v \in \mathcal{A}$$

for some $M > 0$. Then, the attractor is a subset of a Lipschitz manifold given as a graph over $P_N H$ for some N .

The proof of this result follows from a similar argument to the one developed for the proof of Proposition 5.1 below, so we omit it here.

Now consider the embedded vector field on $X = L\mathcal{A}$

$$\dot{x} = h(x) = LGL^{-1}(x), \quad x \in X.$$

As remarked in the Introduction, we would like the inverse of the embedding L to be as smooth as possible and to obtain as much regularity as we can for \mathcal{G} . However, in general, the regularity of \mathcal{G} is determined by the regularity of the linear term A , which can be related to the smoothness of functions on the attractor \mathcal{A} . For example, it follows from the standard interpolation inequality

$$\|Au - Av\| \leq \|u - v\|^{1-(1/r)} \|A^r(u - v)\|^{1/r}, \quad \text{for } u, v \in \mathcal{A}, \quad (11)$$

that, if \mathcal{A} is bounded in $D_H(A^r)$, A is Hölder continuous on \mathcal{A} . In this way, the continuity of F on \mathcal{A} can be deduced from the regularity of solutions on the attractor.

As an example of how one can develop this approach, Foias and Temam [14] showed that, in the two dimensional case, the solutions of the Navier-Stokes equations are analytic in time and proved that the attractor is bounded in $D_H(A^{1/2}e^{\tau A^{1/2}})$. Now, if $u \in D_H(A^{1/2}e^{\tau A^{1/2}})$, then there exists a uniform constant $M > 0$, such that $\|A^{1/2}e^{\tau A^{1/2}}u\|^2 < M$. Hence, $\|A^k u\|^2 \leq M'(4k)!/(2\tau)^{4k}$, where M' is a constant depending uniquely on M . It follows from (11), that

$$\|A(u - v)\| \leq \left[\frac{M'(4j)!}{(2\tau)^{4j}} \right]^{1/2j} \|u - v\|^{1-1/j}.$$

If we minimise the right-hand side over all possible choices of j , we obtain that $A : \mathcal{A} \rightarrow H$ is 2-log-Lipschitz (see [31] for example).

This result relies only on the smoothness of solutions. But one can do much better by making use of the underlying equation. Indeed, Kukavica [20] used the structure of the differential equation (3) and far less restrictive conditions on \mathcal{A} than above to show that $A^{1/2} : \mathcal{A} \rightarrow H$ is 1/2-log-Lipschitz. We briefly outline his argument, which was primarily developed to study the problem of backwards uniqueness for nonlinear equations with rough coefficients, and then show that it can be used to prove that $A : \mathcal{A} \rightarrow H$ is 1-log-Lipschitz.

In what follows we will consider the same equation as in Section 2

$$\frac{du}{dt} + Au = F(u). \quad (12)$$

However, here, we will assume that $\alpha = 1/2$ such that the nonlinear term F is locally Lipschitz from $D_H(A^{1/2})$ into H , i.e.

$$\|F(u) - F(v)\| \leq K(R)\|A^{1/2}(u - v)\|, \quad \text{for all } u, v \in D_H(A^{1/2}), \quad (13)$$

with $\|A^{1/2}u\|, \|A^{1/2}v\| \leq R$, where K is a constant depending only on R . Moreover, we assume that the maximal invariant set \mathcal{A} is bounded in $D_H(A^{1/2})$. The argument that follows is simple – the key observation is that the result is sufficiently abstract that one can make a variety of choices of H (e.g. we will take $H = L^2$ and $H = H^1$).

Let $u(t)$ and $v(t)$ be solutions of (12). The equation for the evolution of the difference $w(t) := u(t) - v(t)$ can be expressed as

$$\frac{dw}{dt} + Aw = f, \quad (14)$$

where $f(t) := F(u(t)) - F(v(t))$. Our assumptions imply that

$$\frac{1}{2} \frac{d}{dt} (Aw, w) = (w_t, Aw) = -(Aw, Aw) + (f, Aw) \quad (15)$$

and

$$\frac{1}{2} \frac{d}{dt} (Aw, Aw) = (w_t, A^2w) = -(Aw, A^2w) + (f, A^2w). \quad (16)$$

Moreover, it follows from (13) that

$$\|f\| \leq \|F(u) - F(v)\| \leq K(\|A^{1/2}u\| \|A^{1/2}v\|) \|A^{1/2}w\| \leq K_1 \|A^{1/2}w\| \quad (17)$$

and, consequently,

$$\operatorname{Re}(f, Aw) \geq -K_2 \|w\| \|A^{1/2}w\| \quad (18)$$

for some $K_1, K_2 \geq 0$.

Under these mild regularity assumptions, Kukavica [20] proved the backward uniqueness property, i.e. if $w : [T_0, 0] \rightarrow H$ is a solution of (14), then $w(0) = 0$ implies that $w(t) = 0$ for all $t \in [T_0, 0]$. His approach consists in establishing upper bounds for the log-Dirichlet quotient

$$\tilde{Q}(t) = \frac{(Aw(t), w(t))}{\|w(t)\|^2 \left(\log \frac{M^2}{\|w(t)\|^2} \right)},$$

where M is a sufficiently large constant. This quantity is a variation of the standard Dirichlet quotient $Q(t) = \|A^{1/2}u\|^2 / \|u\|^2$ (see [25], [3] for details). Kukavica showed that, for equations of the form of (14), the log-Dirichlet quotient is bounded for all $t \geq 0$ and, as an application of this result, stated the following theorem.

Theorem 4.2 ((After Kukavica [20])). *Under the above assumptions on the equation (12) with $F : D_H(A^{1/2}) \rightarrow H$ and $\mathcal{A} \subset D_H(A^{1/2})$, there exists a constant $C > 0$ such that*

$$\|A^{1/2}(u - v)\|^2 \leq C \|u - v\|^2 \log(M^2 / \|u - v\|^2), \quad \text{for all } u, v \in \mathcal{A}, u \neq v,$$

where $M = 4 \sup_{u \in \mathcal{A}} \|u\|$.

We give a quick summary of Kukavica's proof, filling in some details in the closing part of the argument.

of Theorem 4.2. Let

$$L(\|w\|) = \log \frac{M^2}{\|w\|^2},$$

where M is any constant such that

$$M \geq 4 \sup_{u_0 \in \mathcal{A}} \|u_0\|.$$

Note that $L(\|w(t)\|) \geq 1$ for all $t \in [0, T_0]$. For $t \in [0, T_0]$, denote $\tilde{L}(t) = L(\|w(t)\|)$. Define the log-Dirichlet quotient as

$$\tilde{Q}(t) = \frac{Q(t)}{L(\|w\|)} = \frac{\|A^{1/2}w\|^2}{\|w\|^2 L(\|w\|)} = \frac{\|A^{1/2}w\|^2}{\|w\|^2 \tilde{L}(t)}$$

where $Q(t) = \|A^{1/2}w\|^2/\|w\|^2$.

Using (15) and (16), Kukavica [20] showed in the proof of his Theorem 2.1 that

$$\tilde{Q}'(t) + K_3\tilde{Q}(t)^2 \leq K_4, \quad (19)$$

for $0 < K_3 < 1$ and $K_4 \geq 4K_1^4/4(1 - K_3) \geq 0$. Applying a variant of Gronwall's inequality³ proved in Temam [35, Lemma 5.1] to (19), we obtain that there exists T such that

$$\tilde{Q}(t) \leq C(K_3, K_4), \quad \text{for all } t \geq T,$$

where $C(K_3, K_4)$ is a constant independent of $\tilde{Q}(0)$.

Now, consider $u_0, v_0 \in \mathcal{A}$. Since solutions in the attractor exist for all time, we know there exists $t \geq T$ such that $u_0 = S(t)u(-t)$ and $v_0 = S(t)v(-t)$ with $u_0 \neq v_0$. So, $u(-t) \neq v(-t)$. Moreover, $\tilde{Q}(-t) < \infty$ implies that $\tilde{Q}(0) \leq C(K_3, K_4)$. Hence,

$$\sup_{u_0, v_0 \in \mathcal{A}, u_0 \neq v_0} \tilde{Q}(t) \leq C(K_3, K_4).$$

□

We now show that this result can be used to show that $A : \mathcal{A} \rightarrow H$ is 1-log-Lipschitz. Write $w = u - v$. If (17) and (18) hold with $H = L^2$, then there exists a constant $C_0 > 0$ such that

$$\|A^{1/2}w\|_{L^2}^2 \leq C_0\|w\|_{L^2}^2 \log(M_0^2/\|w\|_{L^2}^2), \quad (22)$$

where

$$M_0 \geq 4 \sup_{u \in \mathcal{A}} \|u\|_{L^2}.$$

This is the result of Kukavica [20, Theorem 3.1] for the 2D Navier-Stokes equation.

³Lemma 5.1 (p167 in Temam [35]): *Let y be a positive absolutely continuous function on $(0, \infty)$, which satisfies*

$$y' + \gamma y^p \leq \delta \quad (20)$$

with $p > 1$, $\gamma > 0$, $\delta \geq 0$. Then, for $t \geq 0$

$$y(t) \leq \left(\frac{\delta}{\gamma}\right)^{1/p} + (\gamma(p-1)t)^{-1/(p-1)}. \quad (21)$$

Now assume that \mathcal{A} is bounded in $D_H(A)$. If (17) and (18) hold with $H = D_{L^2}(A^{1/2})$, then there exists a constant $C_1 > 0$ such that

$$\|Aw\|_{L^2}^2 \leq C_1 \|A^{1/2}w\|_{L^2}^2 \log(M_1^2/\|A^{1/2}w\|_{L^2}^2) \quad (23)$$

where

$$M_1 \geq 4 \sup_{u \in \mathcal{A}} \|A^{1/2}u_0\|_{L^2}.$$

So,

$$\|Aw\|_{L^2}^2 \leq C_0 C_1 \|w\|_{L^2}^2 \log(M_0^2/\|w\|_{L^2}^2) \log(M_1^2/\|w\|_{H^1}^2).$$

Since $\|w\|_{L^2} \leq \|w\|_{H^1}$,

$$\|Aw\|_{L^2}^2 \leq C_0 C_1 \|w\|_{L^2}^2 \log(M_0^2/\|w\|_{L^2}^2) \log(M_1^2/\|w\|_{L^2}^2).$$

One can choose M_0 and M_1 such that $M_0 \leq M_1$. Hence,

$$\|Aw\|_{L^2} \leq C \|w\|_{L^2} \log(M_1^2/\|w\|_{L^2}^2), \quad (24)$$

where $C = \sqrt{C_0 C_1}$.

Corollary 4.3. *Under the above assumptions on the equation (12), if \mathcal{A} is bounded in $D_H(A)$, then there exists a constant $C > 0$ such that*

$$\|A(u - v)\| \leq C \|u - v\| \log(M_1^2/\|u - v\|^2), \quad \text{for all } u, v \in \mathcal{A}, u \neq v,$$

where $M_1 \geq 4 \sup_{u \in \mathcal{A}} \|A^{1/2}u\|$.

Unfortunately, this result is not strong enough to prove the existence of a smooth finite-dimensional invariant manifold that contains the attractor. Hence, it would be interesting to know whether, in such a general setting, the 1-log-Lipschitz continuity, obtained for the linear term A , is sharp or if it can be improved. Nevertheless, one can use Corollary 4.3 to show that there exists a family of approximating Lipschitz manifolds \mathcal{M}_N , given as Lipschitz graphs defined over a N -dimensional spaces, such that the global attractor \mathcal{A} associated with equation (12) lies within an exponentially small neighbourhood of \mathcal{M}_N without a making use of the squeezing property.

5 Family of Lipschitz manifolds

Using the inequality (24) obtained in Section 4, one can show, for a wide class of parabolic equations, the existence of a family of Lipschitz manifolds \mathcal{M}_N such that

$$\text{dist}(\mathcal{M}_N, \mathcal{A}) \leq Ce^{-k\lambda_{N+1}},$$

where \mathcal{M}_N is an N -dimensional manifold and C and k are positive constants. We obtain this result without appealing to the squeezing property, on which many constructions in the theory of inertial manifolds rely (see Foias, Manley and Temam [11], for example).

Proposition 5.1. *Suppose that, for some $C > 0$,*

$$\|Aw\|_{L^2} \leq C\|w\|_{L^2} \log(M_1^2/\|w\|_{L^2}^2), \quad (25)$$

where $w = u - v$ for $u, v \in \mathcal{A}$. Then, under the above conditions on equation (14), for each $n > 0$, there exists a Lipschitz function $\Phi_n : P_n H \rightarrow Q_n H$,

$$\|\Phi_n(p_1) - \Phi_n(p_2)\|_{L^2} \leq \|p_1 - p_2\|_{L^2} \quad \text{for all } p_1, p_2 \in P_n H,$$

such that \mathcal{A} lies within a $2M_1^2 e^{-\lambda_{n+1}/\sqrt{2}C}$ -neighbourhood of the graph Φ_n ,

$$\mathbf{G}[\Phi_n] = \{u \in H : u = p + \Phi_n(p), p \in P_n H\}.$$

Note that the method developed in this proof can also be used to prove Proposition 4.1.

Proof. Let $w = u - v$, for $u, v \in \mathcal{A}$. We can split $w = P_n w + Q_n w$, and observe that

$$\begin{aligned} \|Aw\|_{L^2}^2 &= \|A(P_n w + Q_n w)\|_{L^2}^2 = \|A(P_n w)\|_{L^2}^2 + \|A(Q_n w)\|_{L^2}^2 \\ &\geq \lambda_{n+1}^2 \|Q_n w\|_{L^2}^2. \end{aligned}$$

It follows from (25) that

$$\begin{aligned} \|Aw\|_{L^2}^2 &\leq C^2 \|w\|_{L^2}^2 \left(\log(M_1^2/\|w\|_{L^2}^2) \right)^2 \\ &\leq C^2 (\|P_n w\|_{L^2}^2 + \|Q_n w\|_{L^2}^2) \left(\log(M_1^2/\|Q_n w\|_{L^2}^2) \right)^2. \end{aligned}$$

Since $\log (M_1^2/\|Q_n w\|_{L^2}^2) > 1$,

$$\frac{\lambda_{n+1}^2 \|Q_n w\|_{L^2}^2}{\left(\log (M_1^2/\|Q_n w\|_{L^2}^2)\right)^2} \leq C^2 \|P_n w\|_{L^2}^2 + C^2 \|Q_n w\|_{L^2}^2$$

Consider a subset X of \mathcal{A} that is maximal for the relation

$$\|Q_n(u - v)\|_{L^2} \leq \|P_n(u - v)\|_{L^2} \quad \text{for all } u, v \in X. \quad (26)$$

Note that if the P_n components of u and v agree, so that $P_n u = P_n v$, then $Q_n u = Q_n v$. Hence, for every $u \in X$, we can define uniquely $\phi_n(P_n u) = Q_n u$ such that $u = P_n u + \phi_n(P_n u)$. Moreover, it follows from (26) that

$$\|\phi_n(p_1) - \phi_n(p_2)\|_{L^2} \leq \|p_1 - p_2\|_{L^2} \quad \text{for all } p_1, p_2 \in P_n X.$$

Standard results (see Wells and Williams [36], for example) allow one to extend ϕ_n to a function $\Phi_n : P_n H \rightarrow Q_n H$, that satisfies the same Lipschitz bound.

Now, if $u \in \mathcal{A}$ but $u \notin X$, it follows that

$$\|Q_n(u - v)\|_{L^2} \geq \|P_n(u - v)\|_{L^2},$$

for some $v \in X$. Thus, if $w = u - v$, then

$$\frac{\lambda_{n+1}^2 \|Q_n w\|_{L^2}^2}{\left(\log (M_1^2/\|Q_n w\|_{L^2}^2)\right)^2} \leq 2C^2 \|Q_n w\|_{L^2}^2.$$

Hence,

$$\|Q_n w\|_{L^2}^2 \leq M_1^2 e^{-\lambda_{n+1}/\sqrt{2}C},$$

which implies that

$$\begin{aligned} \|w\|_{L^2}^2 &= \|P_n w\|_{L^2}^2 + \|Q_n w\|_{L^2}^2 \leq 2\|Q_n w\|_{L^2}^2 \\ &\leq 2M_1^2 e^{-\lambda_{n+1}/\sqrt{2}C}. \end{aligned}$$

Therefore,

$$\text{dist}(u, \mathbf{G}[\Phi_n]) \leq 2M_1^2 e^{-\lambda_{n+1}/\sqrt{2}C}. \quad (27)$$

□

A similar statement would hold if one used the inequality (22) obtained by Kukavica in [20], involving $A^{1/2}$, rather than (25) that considers A . However, one would obtain a worse exponent in (27), since λ_{n+1} would be replaced by $\lambda_{n+1}^{1/2} \leq \lambda_{n+1}$.

To illustrate this result, we consider the incompressible Navier-Stokes equations

$$\begin{aligned}\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= F, \\ \nabla \cdot u &= 0,\end{aligned}$$

with periodic boundary conditions on $\Omega = [0, 2\pi]^2$ and initial condition $u(x, 0) = u_0(x)$. Here $u(x, t)$ is the velocity vector field, $p(x, t)$ the pressure scalar function, ν the kinematic viscosity and $F(x, t)$ represents the volume forces that are applied to the fluid. We restrict ourselves to the space-periodic case for simplicity. Let \mathcal{H} be the space of all the C^∞ periodic divergence-free functions that have zero average on Ω . Let H be the closure of \mathcal{H} with scalar product $(\cdot, \cdot)_{L^2}$ and norm $\|\cdot\|_{L^2}$, and let V be similarly the closure of \mathcal{H} with scalar product $(\cdot, \cdot)_{H^1}$ and norm $\|\cdot\|_{H^1}$. Let A be the Stokes operator defined by

$$Au = -\Delta u,$$

for all u in the domain $D(A)$ of A in H . Now consider the Navier-Stokes equations written in its functional form

$$\frac{du}{dt} + \nu Au + B(u, u) = F, \quad (28)$$

using the operator A and the bilinear operator B from $V \times V$ into V' defined by

$$(B(u, v), w) = b(u, v, w), \quad \text{for all } u, v, w \in V.$$

If $F \in H$ is independent of time, then the equation (28) possesses a global attractor

$$\mathcal{A} = \left\{ u_0 \in H : S(t)u_0 \text{ exists for all } t \in \mathbb{R}, \sup_{t \in \mathbb{R}} \|S(t)u_0\|_{L^2_{\text{per}}(\Omega)} < \infty \right\},$$

where $S(t)u_0$ denotes a solution starting at u_0 on its maximal interval of existence (cf. Constantin and Foias [6]). Under these assumptions, the difference of solutions $w = u - v$ will satisfy

$$\frac{dw}{dt} + \nu Aw = -[B(w, u) + B(v, w)].$$

So, in this case we use Kukavica's Theorem with $f = -[B(w, u) + B(v, w)]$. Note that

$$\|f\|_{H^1} \leq K_1 \|A^{1/2} w\|_{H^1},$$

and consequently

$$\operatorname{Re}(f, w) \geq -K_2 \|w\|_{H^1} \|A^{1/2} w\|_{H^1}.$$

Therefore, one can apply Proposition 5.1 to the two dimensional Navier-Stokes equation with forcing $F \in L^2$ to show the existence of a family of approximate inertial manifolds of exponential order.

6 Lipschitz Deviation and Embedding Theorem

The existence of a family of approximating Lipschitz manifolds for a dissipative equation of the form of (12) implies that the global attractor \mathcal{A} has zero Lipschitz deviation, a concept that we will define below. In this section, we obtain an embedding of an attractor \mathcal{A} into \mathbb{R}^N that has Hölder continuous inverse, and whose exponent can be made arbitrarily close to one by choosing an embedding space of sufficiently high dimension.

In 1999, Hunt and Kaloshin [17] found an explicit upper bound for the Hölder exponent α in (2), based on the thickness exponent. Later, Olson and Robinson [27] introduced a variation of this quantity that measures how well a compact set X in a Hilbert space H can be approximated by graphs of Lipschitz functions (with prescribed Lipschitz constant) defined over a finite-dimensional subspace of H .

Definition 6.1. (*Olson and Robinson [27]*) Let X be a compact subset of a real Hilbert space H . Let $\delta_m(X, \epsilon)$ be the smallest dimension of a linear subspace $U \subset H$ such that

$$\operatorname{dist}(X, \mathbf{G}_U[\varphi]) < \epsilon,$$

for some m -Lipschitz function $\varphi : U \rightarrow U^\perp$, i.e.

$$\|\varphi(u) - \varphi(v)\| \leq m \|u - v\| \quad \text{for all } u, v \in U,$$

where U^\perp is orthogonal complement of U in H and $\mathbf{G}_U[\varphi]$ is the graph of φ over U :

$$\mathbf{G}_U[\varphi] = \{u + \varphi(u) : u \in U\}.$$

The m -Lipschitz deviation of X , $\text{dev}_m(X)$, is given by

$$\text{dev}_m(X) = \limsup_{\epsilon \rightarrow 0} \frac{\log \delta_m(X, \epsilon)}{-\log \epsilon}.$$

(Since this quantity is bounded and non-increasing in m , the limit as m tends to infinity exists and is equal to the infimum. Indeed, Pinto de Moura and Robinson [28] define the *Lipschitz deviation* of X , $\text{dev}(X)$, via $\text{dev}(X) = \lim_{m \rightarrow \infty} \text{dev}_m(X)$.) Just as in [28], we show that the existence of a family of approximating Lipschitz manifolds, such as that provided by Proposition 5.1, implies that the associated global attractor have zero Lipschitz deviation.

Let $\epsilon_n = 2M_1^2 e^{-\lambda_{n+1}/\sqrt{2}C}$. It follows from Proposition 5.1 that the global attractor \mathcal{A} is contained in an ϵ_n -neighbourhood of a finite-dimensional Lipschitz manifold \mathcal{M} , defined as a graph of $\Phi : PH \rightarrow QH$, with

$$|\Phi(p_1) - \Phi(p_2)| \leq |p_1 - p_2| \quad \text{for all } p_1, p_2 \in PH.$$

Hence, $\delta_1(\mathcal{A}, \epsilon_n) = n$ and

$$\limsup_{n \rightarrow \infty} \frac{\log \delta_1(\mathcal{A}, \epsilon_n)}{-\log \epsilon_n} = \limsup_{n \rightarrow \infty} \frac{\log n}{\sigma \lambda_{n+1} - \log c_0} = 0.$$

Therefore, the global attractor \mathcal{A} for a dynamical system generated by a partial differential equation of the form (17) has $\text{dev}_1(\mathcal{A}) = 0$. Consequently, for a wide class of parabolic equations that satisfy Proposition 5.1, one can apply the following abstract embedding result due to Olson and Robinson [27].

Theorem 6.2 ((Olson and Robinson [27])). *Let \mathcal{A} be a compact subset of a real Hilbert space H with box-counting dimension d and zero Lipschitz deviation. Let $N > 2d$ be an integer and let θ be a real number with*

$$0 < \theta < 1 - \frac{2d}{N}. \tag{29}$$

Then for a prevalent set of linear maps $L : H \rightarrow \mathbb{R}^N$ there exists a $C > 0$ such that

$$C|L(x) - L(y)|^\theta \geq \|x - y\| \quad \text{for all } x, y \in \mathcal{A};$$

in particular these maps are injective on \mathcal{A} .

Note that the Lipschitz deviation is used to bound explicitly the Hölder exponent of the inverse of a linear map L restricted to the image of \mathcal{A} . Therefore, in our case, we can obtain embeddings of \mathcal{A} into \mathbb{R}^N that have a Hölder continuous inverse whose exponent is arbitrarily close to one by taking N sufficiently large.

7 Conclusion

In this paper, we studied conditions under which the global attractor \mathcal{A} is a subset of a Lipschitz manifold given as a graph over a finite-dimensional eigenspace of the linear term A . Then, we showed that, since the linear term of a wide class of dissipative partial differential equations is 1-Log-Lipschitz continuous, the associated global attractor \mathcal{A} lies within a small neighbourhood of a finite-dimensional Lipschitz manifold. Consequently, we are able to obtain linear embeddings of the attractor into \mathbb{R}^N , whose inverse is Hölder continuous with exponent arbitrarily close to one by choosing N sufficiently large.

The existence of a system of ordinary differential equation whose asymptotic behavior reproduces the dynamics on an arbitrary finite-dimensional global attractor remains an interesting open problem. Nevertheless, if we are able to show that there exist exponents $\eta > 0$ and $\gamma > 0$ such that the vector field on the attractor \mathcal{A} is η -log-Lipschitz and the inverse of linear embedding $L : H \rightarrow \mathbb{R}^N$ is γ -log-Lipschitz when restricted to $L\mathcal{A}$, then we will obtain an embedded equation $\dot{x} = h(x)$ with unique solution, provided $\eta + \gamma \leq 1$.

Robinson and Olson [27] showed that, for any $\gamma > 3/2$, we can choose N large enough to obtain a γ -log-bi-Lipschitz embedding into \mathbb{R}^N . However, this lower bound for the exponent γ is too big to ensure uniqueness of solutions. It follows, however, from results obtained in [29] that the exponent γ cannot be made smaller than $1/2$. Therefore,

- we would like to improve the exponent 1 in Corollary 4.3 and
- we would like to reduce the exponent γ of the logarithmic term in [27].

Finally note that if the result for A is optimal, then we need a bi-Lipschitz embedding to guarantee uniqueness of solutions. However, that Romanov [33] obtained a better regularity result for the vector field, than we obtained for the linear term A , suggests that it may be possible to improve the logarithmic exponent in our result.

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